Duality index of oriented regular hypermaps

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Abstract

By adapting the notion of chirality group, the duality group of $\mathcal{H}$ can be defined as the the minimal subgroup $D(\mathcal{H}) \trianglelefteq Mon(\mathcal{H})$ such that $\mathcal{H}/D(\mathcal{H})$ is a self-dual hypermap (a hypermap isomorphic to its dual). Here, we prove that for any positive integer d, we can find a hypermap of that duality index (the order of $D(\mathcal{H})$), even when some restrictions apply, and also that, for any positive integer k, we can find a non self-dual hypermap such that $|Mon(\mathcal{H})|/d = k$. This k will be called the *duality coindex* of the hypermap.

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1 Operations on hypermaps

Topologically, a *map* is a cellular embedding of a connected graph into a closed connected surface. We can generalize this notion if, instead of graphs, we use hypergraphs, allowing each (hyper)edge to be adjacent to more than two (hyper)vertices. By this process, we can construct a more general structure: the *hypermap*. Usually, hypermaps are represented by cellular embeddings of bipartite maps (following the Walsh correspondence between bipartite maps and hypermaps [9]) or by cellular embeddings of connected trivalent graphs. In this last representation (James representation [5]) we label each face 0, 1 or 2 so that each edge of the graph is incident to two faces carrying different labels. Those faces correspond to hypervertices, hyperedges or hyperfaces, depending on the label they carry.

The topological definitions of maps and hypermaps have a combinatorial translation. A map can be understood as a transitive permutation representa-

tion $\Gamma \to \mathrm{Sym}\, F$ of the group

$$\Gamma = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = (r_2 r_0)^2 = 1 \rangle = V_4 * C_2$$

on a set F representing its flags (the cells of the barycentric subdivision of the map); and a hypermap can be regarded as a transitive permutation representation $\Delta \to \mathrm{Sym}\,\Omega$ of the group

$$\Delta = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1 \rangle \cong C_2 * C_2 * C_2,$$

on a set Ω representing its hyperflags. Similarly, an oriented hypermap (without boundary) can be regarded as a transitive permutation representation of the subgroup

$$\Delta^+ = \langle \rho_0, \rho_1, \rho_2 \mid \rho_0 \rho_1 \rho_2 = 1 \rangle = \langle \rho_0, \rho_2 \mid - \rangle$$

of index 2 in Δ (a free group of rank 2) consisting of the elements of even word-length in the generators r_i, where $\rho_0 = r_1 r_2$, $\rho_1 = r_2 r_0$ and $\rho_2 = r_0 r_1$. In the case of hypermaps, the hypervertices, hyperedges and hyperfaces (i-dimensional constituents for $i = 0, 1, 2$) are the orbits of the dihedral subgroups $\langle r_1, r_2 \rangle$, $\langle r_2, r_0 \rangle$ and $\langle r_0, r_1 \rangle$, and in the case of oriented hypermaps they are the orbits of the cyclic subgroups $\langle \rho_0 \rangle$, $\langle \rho_1 \rangle$ and $\langle \rho_2 \rangle$, with incidence given by nonempty intersection in each case. The local orientation around each hypervertex, hyperedge or hyperface is determined by the cyclic order of the corresponding cycle of ρ_0, ρ_1 or ρ_2.

Operations on topological maps were first studied by S. Wilson [10] but were later extended to hypermaps, following a more algebraic approach [8, 4]. If $\mathcal{H}$ is a hypermap corresponding to a permutation representation $\theta : \Delta \to \mathrm{Sym}\,\Omega$, and if α is an automorphism of Δ, then $\alpha^{-1} \circ \theta : \Delta \to \mathrm{Sym}\,\Omega$ corresponds to a hypermap $\mathcal{H}^\alpha$. Therefore, an *operation* on hypermaps is any transformation of hypermaps induced by a group automorphism of Δ. The hypervertices, hyperedges and hyperfaces of $\mathcal{H}^\alpha$ are, respectively, the orbits of $\langle r_1^\alpha, r_2^\alpha \rangle$, the orbits of $\langle r_2^\alpha, r_0^\alpha \rangle$ and the orbits of $\langle r_0^\alpha, r_1^\alpha \rangle$ on Ω. If α is an inner automorphism then $\mathcal{H}^\alpha \cong \mathcal{H}$ for all $\mathcal{H}$, so we have an induced action of the outer automorphism group $\mathrm{Out}\,\Delta = \mathrm{Aut}\,\Delta / \mathrm{Inn}\,\Delta$ as a group Φ of operations on isomorphism classes of hypermaps. This action is faithful, as shown by L. James [4]. The same can be said about oriented hypermaps, with $\mathrm{Out}\,\Delta^+$ acting as a group Φ^+ of operations. L. James [4] also proved that $\mathrm{Out}\,\Delta^+ \cong GL(2, \mathbb{Z}_2)$, a very important result for the classification of all operations on oriented hypermaps (see [7] for details).

2 Algebraic Hypermaps

Δ and Δ^+ are, respectively, the full automorphism group and the orientation-preserving automorphism group of the universal hypermap $\widetilde{\mathcal{H}}$. This hypermap

is called *universal* because any hypermap is the quotient of $\widetilde{\mathcal{H}}$ by some subgroup $H \leq \Delta$, known as the *hypermap subgroup* (which is unique up to conjugacy). If $H \trianglelefteq \Delta$, we say that the hypermap is *regular* since, when this occurs, the hypermap has the highest possible number of symmetries. A regular hypermap can be represented, algebraically, by a four-tuple $\mathcal{H} = (\Delta/H, h_0, h_1, h_2)$ where $h_0^2 = h_1^2 = h_2^2 = 1$ and $\langle h_0, h_1, h_2 \rangle = \Delta/H$, the *monodromy group*, $Mon(\mathcal{H})$, of the hypermap. Similarly, an oriented regular hypermap can be regarded as a triple $\mathcal{H}^+ = (\Delta^+/H, x, y)$, with $\Delta^+/H = \langle x, y \rangle$ being the monodromy group of the oriented regular hypermap. From a topologically point of view, x can be interpreted as the permutation that cyclic permutes the hyperdarts (oriented hyperedges) based on the same hypervertex, and y the permutation that cyclic permutes the hyperdarts based on the same hyperface, according to the chosen orientation. An oriented hypermap is called *chiral* if it is not invariant under the operation that reflects the oriented hypermap, inverting its orientation (which is the same as saying that $\mathcal{H}^+$ admits no orientation-reversing automorphism). If $(xy)^2 = 1$, $\mathcal{H}^+$ is a map.

3 The Duality Group

Our aim is to study what we will call the *duality group* of a hypermap. Some work has been done on chirality groups [2] and there is no reason not to extend that notion to duality or other hypermap operations. These operations, as we have mentioned before, come from outer automorphisms of Δ and by choosing the right group Δ^*, containing Δ, we can look at duality as the result of sending a hypermap subgroup to its conjugate in Δ^*. To build this group, we should add an element t, of order 2, transposing r_0 and r_2 and fixing r_1. Hence, we can define Δ^* in the following way:

$$\Delta^* = \Delta \rtimes C_2 = \langle r_0, r_1, r_2, t : r_i^2 = t^2 = 1, r_0^t = r_2, r_1^t = r_1 \rangle$$

This also means that Δ is a normal subgroup of index 2 of Δ^*. Therefore, each conjugacy class of subgroups $H \leq \Delta$ is either a Δ^*-conjugacy class (if the hypermap $\mathcal{H}$ is self-dual, which occurs when $\mathcal{H} \cong \mathcal{H}^t$) or paired with another Δ-conjugacy class, containing H^t (if the hypermap $\mathcal{H}$ is not self-dual). This last observation is a general one and it is true for every kind of hypermap. However, we will only deal with regular hypermaps and these have normal subgroups as hypermap subgroups, which means that H is conjugate only to itself in Δ. So, if a hypermap is self-dual, the group H is invariant under that specific outer automorphism of Δ (conjugation in Δ^*).

Theorem 3.1. *Let N be a normal subgroup of Δ and let $G = \Delta/N$. Then the following are equivalent:*

i) $N^t = N$

ii) N *is normal in* Δ^* $\square$

Because

$$\Delta^* = \langle r_0, r_1, r_2, t : r_i^2 = t^2 = 1, r_0^t = r_2, r_1^t = r_1 \rangle =$$

$$= \langle r_0, r_1, t : r_0^2 = r_1^t = t^2 = 1,\ r_1^t = r_1 \rangle = \langle r_1, t \rangle * \langle r_0 \rangle \cong V_4 * C_2 \cong \Gamma$$

we can build a functor from hypermaps ($H \leq \Delta$) to maps ($H \leq \Delta^* \cong \Gamma$) and, depending on the chosen isomorphism between Δ^* and Γ, this is the Walsh functor [9], representing a hypermap as a bipartite map or one of its duals.

If $\mathcal{H}$ is a regular hypermap with hypermap subgroup H then H is normal in Δ. The largest normal subgroup of Δ^* contained in H is the group $H_\Delta = H \cap H^t$ and the smallest normal subgroup of Δ^* containing H is the group $H^\Delta = HH^t$. These correspond, respectively, to the smallest self-dual hypermap that covers $\mathcal{H}$, and the largest self-dual hypermap that is covered by $\mathcal{H}$.

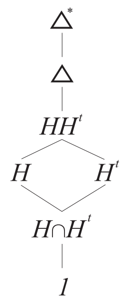

Figure 1: H_Δ and H^Δ.

Like chirality, the duality operation is an operation of order 2 (see [7] for the complete classification of hypermap operations of finite order) and some of the results that were stated for chirality and chirality groups [2] also work here, with similar proves. Whenever this is case, we will not give a demonstration of the result since the reader can easily adapt the one available in [2]. The following proposition is a good example of what we have just mentioned:

Proposition 3.1. *The groups* H^Δ/H, H/H_Δ, H^Δ/H^t *and* H^t/H_Δ *are all isomorphic to each other.* $\square$

This common group will be called the *duality group* $D(\mathcal{H})$ of $\mathcal{H}$ and its order the *duality index* d of $\mathcal{H}$. The duality index is somehow a way to measure how far the hypermap is from being self-dual. If the duality index is 1 then the hypermap is self-dual; and the bigger that index, the more distant the hypermap is from being self-dual.

Proposition 3.2. *The duality group $D(\mathcal{H})$ of a regular hypermap $\mathcal{H}$ is isomorphic to a normal subgroup of the monodromy group $Mon(\mathcal{H})$.*

Proof: The same as in [2], with duality group instead of chirality group. $\square$

Then, another possible way to understand the duality group is to look at it as the minimal subgroup $D(\mathcal{H}) \trianglelefteq Mon(\mathcal{H})$ such that $\mathcal{H}/D(\mathcal{H})$ is a self-dual hypermap. If $D(\mathcal{H}) = Mon(\mathcal{H})$ or, equivalently, $H^{\Delta} = \Delta$ we say that the hypermap has *extreme duality index*.

We have extended Δ to Δ^* by adjoining t such that $t^2 = 1, \quad r_0^t = r_2, \quad r_2^t = r_0, \quad r_1^t = r_1$. Then,

$$x^t = r_2 r_1 = y^{-1}, \quad y^t = r_1 r_0 = x^{-1}.$$

We will denote this kind of duality on oriented regular hypermaps by $\beta - duality$ (chiral-duality). On the other hand, conjugation by $r_1 t$ induces

$$x \mapsto y, \quad y \mapsto x,$$

interchanging generators. This will be called $\alpha - duality$ (orientation-preserving duality). The relationship between the two (α and β) will be dealt briefly at the end of this paper. From now on, to simplify the writing, whenever we refer to *duality* we mean $\alpha - duality$, the one that preserves orientation. An hypermap is called *self-dual* if it is invariant under this duality operation.

From an orientable hypermap we can choose two possible oriented hypermaps. If $\mathcal{H} = (G, r_0, r_1, r_2)$, let $\mathcal{H}^+ = (G^+, x, y)$ be one of the oriented hypermaps associated with $\mathcal{H}$. The duality group $D(\mathcal{H}^+)$ is the minimal normal subgroup of G^+ such that $\mathcal{H}^+/D(\mathcal{H}^+)$ is a self-dual hypermap. It follows that $D(\mathcal{H}^+) \trianglelefteq G^+$ and we say that $\mathcal{H}$ has extreme duality index if $D(\mathcal{H}^+) = G^+$. (Hence, a hypermap has extreme duality index if its duality group is equal to its monodromy group).

4 Duality index

We can now easily prove the following theorem:

Theorem 4.1. *For every $k \in \mathbb{N}$, there is a self-dual oriented regular hypermap with order k.*

Proof: Let G be the cyclic group of order k generated by g. If we take $G = \langle g \rangle$ and $\mathcal{H} = (G, g, g)$, the hypermap with monodromy group G, then there is an automorphism of $\mathcal{H}$ that interchanges the two generators (they are both equal to g, in this case). Hence, the hypermap is self-dual. $\square$

But this last result also means that for every $k \in \mathbb{N}$ there is a hypermap $\mathcal{H} = (G, a, b)$ such that $|G|/d = k$, with d being the duality index of $\mathcal{H}$. We just have to take $G = \langle g \rangle$, as the cyclic group of order k, and the hypermap $\mathcal{H} = (G, g, g)$, as in the previous proof. Because (G, g, g) is self-dual, $d = 1$ and we have $|G|/d = |G| = k$. From now on, we will call $|G|/d$ the *duality coindex* of a hypermap of monodromy group G.

Can we prove a similar theorem as Theorem 4.1 using only hypermaps that are not self-dual (for which $d \neq 1$)? The proof we provide bellow will give the reader not only an affirmative answer but also the presentation of the monodromy groups of those hypermaps.

Theorem 4.2. *If $k \in \mathbb{N}$, there is a non self-dual oriented regular hypermap $\mathcal{H} = (G, a, b)$ with duality coindex k.*

Proof: Given $k \geq 3$, we can choose, by Dirichlet's Theorem, a prime $q \equiv 1 \bmod (k)$. Let $G = \langle g, h | h^q = 1, g^k = 1, h^g = h^u \rangle \cong C_q \rtimes C_k$, where $u \in \mathbb{Z}_q$ has multiplicative order k, $C_q = \langle h \rangle$ and $C_k = \langle g \rangle$. Then, if $h = ab$ and $g = a$, we have:

$$G = \langle a, b | (ab)^2 = a^k = 1, (ab)^a = (ab)^u \rangle$$

The duality group of this hypermap is the smallest normal subgroup N of G such that the assignment $a \mapsto b$, $b \mapsto a$ induces an automorphism of G/N. We obtain this quotient by adding extra relations, substituting a for b and b for a in the original ones.[1] In this case, we just have to add these relations: $b^k = 1$ and $(ba)^b = (ba)^u$.Hence:

$$G/N = \langle (ab)^2 = a^k = b^k = 1, (ab)^a = (ab)^u, (ba)^b = (ba)^u \rangle$$

But $(ab)^a = ba$, so $ba = (ab)^u$, $ab = (ba)^u$. It follows that $ab = (ab)^{u^2}$ or, equivalently, $(ab)^{u^2-1} = 1$. Because $k \geq 3$, we have $u \neq \pm 1 \bmod q \Rightarrow u^2 - 1 \neq 0 \bmod q \Rightarrow (u^2 - 1, q) = 1$. So,

$$(ab)^q = (ab)^{u^2-1} = 1 \Rightarrow ab = 1 \Rightarrow b = a^{-1}.$$

Thus $G/N = \langle a | a^k = 1 \rangle \cong C_k$. Therefore $|G/N| = k$ and, since G is not cyclic, the hypermap $\mathcal{H} = (G, a, b)$ is not self-dual.

If $k = 2$ we take $G = C_6$ generated by the pair $(1, 4)$, with presentation:

$$C_6 = \langle x, y | x^6 = 1, x^4 = y \rangle.$$

Hence, considering N as the duality group,

$$|C_6/N| = |\langle x, y | x^6 = 1, x^4 = y, y^6 = 1, y^4 = x \rangle| = 3.$$

[1]This method will be used several times in the next pages and the group N, in similar contexts, will always mean the smallest normal subgroup N of the monodromy group G such that the interchange of generators induces an automorphism of G/N.

Then, (C_6, x, y) has duality index $\frac{6}{3} = 2$.

For $k = 1$, all we have to do is to choose any hypermap with extreme duality index. $\square$

Now, another question can be asked: for each $d \in \mathbb{N}$, is it possible to find at least one hypermap with that duality index? And can we make some restrictions in the available hypermaps we are allowed to choose? The first question is not difficult to be answered:

Theorem 4.3. *For every $d \in \mathbb{N}$, there is an oriented regular hypermap with duality index equal to d.*

Proof: Let G be the cyclic group of order d generated by g. If we take $G = \langle g \rangle$ and $\mathcal{H} = (G, g, 1)$, the hypermap with monodromy group G, then its duality group must be equal to G, which means that the hypermap has an extreme duality index $|G| = d$. $\square$

Remarks: a) Obviously, $\mathcal{H} = (G, 1, g)$ also works here. In fact, for any duality index, we can always find, not just one, but two hypermaps with that extreme duality index (which is not surprising since these two hypermaps are duals of each other). b) It follows from the proof of this last theorem that for every $n \geq 1$ there is an oriented regular hypermap with cyclic duality group (the monodromy group of the hypermap $\mathcal{H} = (C_n, g, 1)$ with $C_n = \langle g \rangle$).

It is now clear that we can get any duality index using hypermaps that have extreme duality index. Can we achieve the same result only with hypermaps that do not have extreme duality index? Before we answer that question, we need to introduce some results and definitions about direct products of hypermaps.

5 Direct Products and Duality groups

If $\mathcal{H}$ and $\mathcal{K}$ are oriented regular hypermaps with hypermaps subgroups H and $K \leq \Delta^+$, respectively, then:

Definition 5.1. The *least common cover* $\mathcal{H} \vee \mathcal{K}$ and the *greatest common quotient* $\mathcal{H} \wedge \mathcal{K}$ are the oriented regular hypermaps with hypermap subgroups $H \cap K$ and $\langle H, K \rangle = HK$ respectively.

If $\mathcal{H} = (D_1, R_1, L_1)$ and $\mathcal{K} = (D_2, R_2, L_2)$ let $D = D_1 \times D_2$ and the permutations R and L be the ones that act on D induced by the actions $\rho \mapsto R_i$, $\lambda \mapsto L_i$ of Δ^+ on D_1 and D_2. If this action is transitive on D, we call $\mathcal{H} \times \mathcal{K} = (D, R, L)$, the *oriented direct product* of $\mathcal{H}$ and $\mathcal{K}$ with hypermap subgroup $H \cap K$.

Lemma 5.1. [1] *If $\mathcal{H}$ and $\mathcal{K}$ are oriented regular hypermaps, then the following conditions are equivalent:*

i) Δ^+ *acts transitively on* D*;*

ii) $\mathcal{H} \wedge \mathcal{K}$ *is the oriented hypermap, with one dart;*

iii) $HK = \Delta^+$. □

If these conditions are satisfied we say that $\mathcal{H}$ and $\mathcal{K}$ are *oriented orthogonal* and we use the notation $\mathcal{H} \perp \mathcal{K}$. Then, $\mathcal{H} \times \mathcal{K}$ is well defined and isomorphic to $\mathcal{H} \vee \mathcal{K}$ with monodromy group $Mon(\mathcal{H} \times \mathcal{K}) = Mon(\mathcal{H}) \times Mon(\mathcal{K})$. Having in mind that $\mathcal{H}$ has extreme duality index if and only if $HH^d = \Delta^+$, we have, as an important example, the following result:

Lemma 5.2. *$\mathcal{H}$ has extreme duality index* $\Leftrightarrow \mathcal{H} \perp \mathcal{H}^d$. □

Once again, we can adapt one of the theorems for chirality groups [2], writing it in this new context of duality:

Theorem 5.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be oriented regular hypermaps, with hypermap subgroups H and K, such that $\mathcal{K}$ has extreme duality index and covers $\mathcal{H}$. Then the product $\mathcal{L} = \mathcal{K} \times \mathcal{H}^d$ is an oriented regular hypermap with duality group $D(\mathcal{L}) \cong H/K$.*

Proof: The same as in [2], substituting chirality by duality. □

We can now answer the question we have raised at the end of the previous section:

Theorem 5.2. *For every $d \in \mathbb{N}$ there is an oriented regular hypermap with non extreme duality index d.*

Proof: Let K be a normal subgroup of Δ^+ such that $\Delta^+/K = C_{2d}$. Then $\mathcal{K} = (C_{2d}, g, 1)$, with $C_{2d} = \langle g \rangle$, is orientably regular and has extreme duality index. If we take H such that $K \leq H$ and $|H : K| = d$ then $|\Delta^+ : H| = 2$, which means that $H \trianglelefteq \Delta^+$ and $\mathcal{H}$ is orientably regular. Hence, the hypermap $\mathcal{L} = \mathcal{K} \times \mathcal{H}^t$ is orientably regular and

$$|Mon(\mathcal{L})| = |Mon(\mathcal{K})| \cdot |Mon(\mathcal{H}^t)| = 2d \times 2 = 4d.$$

Then, by Theorem 5.1:

$$\mathcal{D}(\mathcal{L}) = H/K,$$

and $|H : K| = d$. $\mathcal{L}$ does not have extreme duality index because $|Mon(\mathcal{L})| = 4d > d$. □

This is not only true for *hypermaps* but also for *maps*:

Theorem 5.3. *For every $d \in \mathbb{N}$ there is an oriented regular map with (non extreme) duality index equal to d.*

Proof: Let $D_{2m} = \langle x, y | x^m = y^2 = (xy)^2 = 1\rangle$ be the dihedral group of order $2m$. If we take $\mathcal{M} = (D_{4d}, x, y)$, then, considering N as before, we will have: $D_{4d}/N \cong D_4$. Therefore, $|N| = 4d/4 = d$. $\square$

Although a map is a special case of a hypermap (when $(xy)^2 = 1$), Theorem 5.3 is not a stronger version of Theorem 5.2, since Theorem 5.2 allows us to get not just hypermaps but *proper* hypermaps (hypermaps that are not maps), which is also an important restriction.

A group is called *strongly self-dual* if for all its generating pairs there is an automorphism of G interchanging them. A good example of one of these groups is the quaternion group. In the next section, we will use a generalization of the quaternion group to find infinite families of proper hypermaps with non extreme duality indexes.

6 Generalized quaternion groups

Definition 6.1. If $w = e^{i\pi n} \in \mathbb{C}$, the matrices:

$$x = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generate a subgroup Q_{2n} of order $4n$ in $GL(2, \mathbb{C})$ with presentation [6]:

$$\langle x, y | x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1}\rangle$$

which is called the *generalized quaternion group*.

As we have proved in Theorem 5.2, we can have a hypermap of any non extreme duality index. However, that proof does not show us the presentation of the monodromy group of any of those hypermaps. Various explicit examples can be obtained using generalized quaternion groups.

Theorem 6.1. *If d is odd or $d \equiv 0$ (mod 4), there is an oriented regular hypermap with generalized quaternion monodromy group, which has a non extreme duality index equal to d.*

Proof: a) d is odd:

Let $n = 2+4k$, $k = 0, 1, 2, \ldots$. If we take G to be the generalized quaternion group of order $4n$ then $|G| = 8 + 16k$ and has presentation:

$$G = \langle x, y | x^{2+4k} = y^2, x^{4+8k} = 1, y^{-1}xy = x^{-1}\rangle.$$

If we take N to be the smallest normal subgroup of G such that the assignment that interchanges the two generators induces an automorphism then G/N (which

is obtain from G adding new relations) is the quaternion group and has order 8. But $|N| = |G|/|G/N|$. Hence, $|N| = 8 + 16k/8 = 2k + 1$(for $k = 0, 1, ...$) From this, we can conclude that for d odd there is a hypermap with monodromy group G and a non extreme duality index (since $|G/N| = 8 \neq 1$) equal to $d = 2k + 1$.

b) $d \equiv 0$ $(mod\ 4)$:

Let $n = 4k$, $k = 1, 2, ...$ If we take G to be the generalized quaternion group of order $4n$ then $|G| = 16k$ and has presentation:

$$G = \langle x, y | x^{4k} = y^2, x^{8k} = 1, y^{-1}xy = x^{-1} \rangle$$

If we take N to be the smallest normal subgroup of G such that the assignment that interchanges the two generators induces an automorphism then

$$G/N = \langle x, y | x^{4k} = y^2, x^{8k} = 1, y^{-1}xy = x^{-1}, y^{4k} = x^2, y^{8k} = 1, x^{-1}yx = y^{-1} \rangle$$

Using the third and sixth relations, we have $(y^{-1}xy)yx = x^{-1}(xy^{-1}) = y^{-1}$. Therefore, applying the first relation $y^2 = x^{4k}$, we have: $y^{-1}xx^{4k}x = y^{-1} \Rightarrow x^{4k+2} = 1$. Then, using the second relation: $x^{4k+2} = x^{8k} \Rightarrow x^{4k-2} = 1$. Hence, $x^{4k+2} = x^{4k-2} = 1 \Rightarrow x^4 = 1$.

From the first relation $x^{4k} = y^2$ we can now conclude that $y^2 = 1$ and, from the fourth one, that $x^2 = 1$. Therefore, the presentation of the group G reduces to $\langle x, y | x^2 = y^2 = (xy)^2 = 1 \rangle$, which defines a Klein 4-group. We have proved, this way, that G/N has order 4. Hence

$$|N| = 16k/4 = 4k \quad , \quad k = 1, 2, ...$$

This means that, for $d \equiv 0$ $(mod\ 4)$, there is a hypermap with monodromy group G has a non extreme duality index (since $|G/N| = 4 \neq 1$) equal to $d = 4k$. □

Corollary 6.1. *Every cyclic group of odd order or of order multiple of 4 can be a duality group of an oriented regular hypermap with non extreme duality index and generalized quaternion monodromy group.*

Proof: In the previous proof $N = \langle x^4 \rangle \cong C_{1+2k}$, in a); and $N = \langle x^2 \rangle \cong C_{4k}$, in b). □

In the proof of the Theorem 6.1, G/N is the quaternion group and any hypermap which has that group as monodromy group is self-dual. But all generating pairs are equivalent under automorphisms of the quaternion group. Then, there is only one (self-dual) hypermap, up to isomorphism, with monodromy group being the quaternion group.

Theorem 6.2. *Let n be odd. Then, the generalized quaternion group*

$$G = \langle x, y | x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle$$

of order $4n$ is the monodromy group of an oriented regular hypermap with extreme duality index.

Proof: If we take N to be the smallest normal subgroup of G such that the assignment that interchanges the two generators induces an automorphism then

$$G/N = \langle x, y | x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1}, y^n = x^2, y^{2n} = 1, x^{-1}yx = y^{-1} \rangle.$$

Hence, we have $x^{-1}yx = y^{-1}$ (last relation) but also $x^{-1} = y^{-1}xy$ (third relation). Therefore, $y^{-1}xyyx = y^{-1} \Rightarrow y^{-1}xy^2x = y^{-1}$. Using the first relation in this last equality, we have $y^{-1}xx^nx = y^{-1} \Rightarrow x^{n+2} = 1$. Let k be the order of x. Then, since $k|(n+2)$ and n is odd, k must also be odd. But from the second relation we also know that $x^{2n} = 1$ and, consequently, $k|2n$. Therefore, $k|n$. If odd k divides n and $n+2$, then $k = 1$ (and we have $x = 1$). Because $y^n = y^2$ and $y^2 = x^n$, we have $y^n = y^2 = 1$. Since n is odd, $y = 1$. Hence, $|G/N| = 1$, which means that the hypermap has extreme duality index. $\square$

Corollary 6.2. *There are infinitely many oriented regular hypermaps with extreme duality index and generalized quaternion group as monodromy group.* $\square$

Every hypermap having the generalized quaternion group (with the presentation given in our definition) as monodromy group has chirality index equal to 1. This can easily be checked because if we want to obtain a reflexible hypermap as a quotient of the original one, we just have to add the following relations to the ones that we already have for the generalized quaternion group: $x^{-n} = y^{-2}$, $x^{-2n} = 1$ and $yx^{-1}y^{-1} = x$ (substituting x by x^{-1} and y by y^{-1} in the original relations) . However, these relations do not change the presentation of the group. Hence, all the theorems above (where the generalized quaternion group appears in the proof) are, in fact, about reflexible (non chiral) hypermaps.

7 Chiral duality

As we have previously noticed, there are two types of duality induced by the following automorphisms of Δ^+:

$$\alpha : x \mapsto y; \quad y \mapsto x,$$
$$\beta : x \mapsto y^{-1}; \quad y \mapsto x^{-1}.$$

Since the automorphisms of Δ^+ which induce them are conjugate in $Aut(\Delta^+)$, both dualities have the same general properties (the groups which arise as $\alpha-$*duality* groups are the same that arise as β-duality groups [7]). Nevertheless, their effect on a *specific* hypermap might be distinct. To make this observation clear to the reader, we will give some examples of hypermaps such that:

a) $|\mathcal{D}_\alpha(\mathcal{H})| \neq |\mathcal{D}_\beta(\mathcal{H})|$

b) $\mathcal{D}_\alpha(\mathcal{H}) \cong \mathcal{D}_\beta(\mathcal{H})$

Examples

a) We can take $\mathcal{H} = (G, x, y)$ with $G = \langle x, y | x^4 = y^4 = 1, \quad xy = y^2x^2\rangle$. $|G| = 20$ (this order can easily be checked using GAP [3]). Then

$$G/N_\alpha = \langle x, y | x^4 = y^4 = 1, \quad xy = y^2x^2, \quad yx = x^2y^2\rangle.$$

Using the two last relations, we have: $xyyx = y^2x^2x^2y^2 \Leftrightarrow xy^2x = 1 \Leftrightarrow y^2 = x^2$. Therefore, $G/N = \langle x | x^4 = 1\rangle$ and $|G/N_\alpha| = 4$. However:

$$G/N_\beta = \langle x, y | x^4 = y^4 = 1, \quad xy = y^2x^2, \quad y^{-1}x^{-1} = x^{-2}y^{-2}\rangle = G.$$

Hence $|G/N_\beta| = 20$. It follows that G is β-self-dual but not α-self-dual.

b) If $\mathcal{H} = (G, x, y) = (A_5, (12345), (123))$ then $D_\alpha(\mathcal{H}) \cong A_5$ because the hypermap has extreme α-duality index. But $\mathcal{H}^\beta = (G, y^{-1}, x^{-1}) = (A_5, (132), (15432))$. Hence, we still have two permutations of different order. This means that the hypermap cannot be β-self-dual and, because A_5 is simple, we can conclude that it must have extreme β-duality index. It follows that $D_\beta(\mathcal{H}) \cong D_\alpha(\mathcal{H}) \cong A_5$.

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